EE120 - Fall'19 - Lecture 15 Notes[1]

Murat Arcak

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The Laplace Transform

Chapter 9 *in* Oppenheim & Willsky

The Laplace transform of a continuous-time signal x is

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt} \tag{1}$$

where s is a complex variable. $X(s)|_{s=j\omega}$ is the Fourier transform.

Recall from Lecture 3:

$$e^{st} \rightarrow \boxed{h(t)} \rightarrow H(s)e^{st} \quad \text{where} \quad H(s) := \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

Thus the transfer function $H(s)$ is the Laplace transform of $h(t)$.

<u>Example 1:</u>

$$\boxed{x(t) = e^{-at}u(t)} \quad \leftrightarrow \quad X(j\omega) = \frac{1}{j\omega + a} \quad \text{if} \quad a > 0 \quad \text{(Lecture 6)}$$

Find the Laplace transform:

$$X(s) = \int_{0}^{\infty} e^{-at}e^{-st}dt.$$

Let σ denote the real part of s ($s = \sigma + j\omega$):

$$\begin{aligned} X(s) &= \int_0^{\infty} e^{-(a+\sigma)t}e^{-j\omega t}dt = \underset{\text{(convergence if } a+\sigma>0)}{\text{Fourier transform of } e^{-(a+\sigma)}u(t)} \\ &= \frac{1}{j\omega + (a+\sigma)} = \frac{1}{s+a} \end{aligned}$$

Therefore,

$$\boxed{X(s) = \frac{1}{s+a} \quad \text{if} \quad \sigma = Re\{s\} > -a.}$$

If $a = 0$ (unit step), Fourier transform doesn't converge, but the Laplace transform does for $Re\{s\} > 0$.

<u>Example 2:</u> $\boxed{x(t) = -e^{-at}u(-t)}$

Using the change of variables $\tau = -t$:

$$\begin{aligned} X(s) &= -\int_{-\infty}^{0} e^{-at}e^{-st}dt \\ &= -\int_{\infty}^{0} e^{(a+s)\tau}(-d\tau) = -\int_0^{\infty} e^{(a+s)\tau}d\tau = -\frac{1}{a+s}e^{(a+s)\tau}|_0^{\infty} \end{aligned}$$

If $Re\{a+s\} < 0$, then $e^{(a+s)\tau} \rightarrow 0$ as $\tau \rightarrow \infty$: $\boxed{= \frac{1}{s+a} \text{ if } Re\{s\} < -a}$

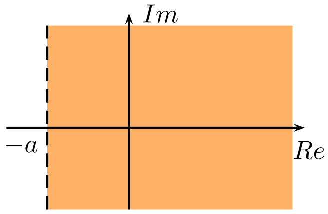
Region of Convergence (ROC)

Note that the Laplace and Fourier transforms are related by

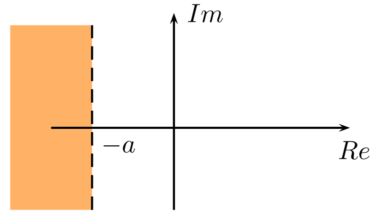
$$\mathcal{L}\{x(t)\} = \mathcal{F}\{x(t)e^{-\sigma t}\} \text{ where } \sigma = Re\{s\}.$$

Thus, the ROC is the set of $s \in \mathbb{C}$ whose real part σ is such that the Fourier integral for $x(t)e^{-\sigma t}$ converges. If the ROC includes the imaginary axis ($\sigma = 0$), then the Fourier transform exists for $x(t)$.

Ex. 1:



Ex. 2:



Example 3: $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$\begin{aligned} X(s) &= \frac{3}{s+2} - \frac{2}{s+1} \\ &= \frac{s-1}{(s+1)(s+2)} \end{aligned} \qquad \begin{aligned} ROC &= \{s|Re\{s\} > -2\} \cap \{s|Re\{s\} > -1\} \\ &= \{s|Re\{s\} > -1\} \end{aligned}$$

Example 4: $x(t) = e^{-\alpha t}u(t)$, α: complex.

$$X(s) = \frac{1}{s+\alpha} \text{ if } Re\{s+\alpha\} > 0, \text{ i.e., if } Re\{s\} > -Re\{\alpha\}.$$

Example 5: $\cos(\omega_0 t)u(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$

From Example 4 the Laplace transform is:

$$\frac{1}{2}\frac{1}{s-j\omega_0} + \frac{1}{2}\frac{1}{s+j\omega_0} = \frac{1}{2}\frac{2s}{s^2+\omega_0^2} = \frac{s}{s^2+\omega_0^2}$$

$$\boxed{\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega_0^2} \quad Re\{s\} > 0} \tag{2}$$

A similar derivation shows:

$$\boxed{\sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{s^2+\omega_0^2} \quad Re\{s\} > 0} \tag{3}$$

Example 6: $e^{-at}cos(\omega_0 t)u(t) = \frac{1}{2}e^{-(a-j\omega_0)t}u(t) + \frac{1}{2}e^{-(a+j\omega_0)t}u(t)$

From Example 4:

$$\frac{1}{2}\frac{1}{s+a-j\omega_0} + \frac{1}{2}\frac{1}{s+a+j\omega_0} = \frac{(s+a)}{(s+a)^2+\omega_0^2}$$

$$\boxed{e^{-at}\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{(s+a)}{(s+a)^2+\omega_0^2} \quad Re\{s\} > -a} \tag{4}$$

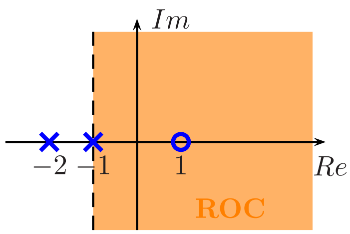
Poles and Zeros of Laplace Transforms

Suppose $X(s)$ is rational (polynomial divided by polynomial):

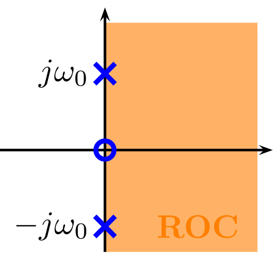
$$X(s) = \frac{N(s)}{D(s)}. \tag{5}$$

Zeros of $X(s)$ are the roots of the numerator $N(s)$, and poles are the roots of the denominator $D(s)$.

Ex. 3:



Ex. 5:



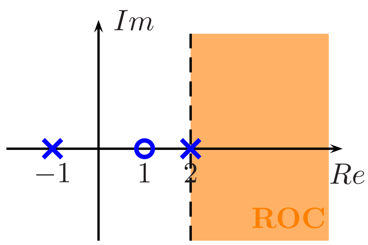
Zeros marked with "**o**", and poles with "**x**".

<u>Example 7:</u>

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

Laplace transform of $\delta(t)$: $\int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$ for all $s \in \mathbb{C}$.

$$\begin{aligned} X(s) &= 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} \\ &= \frac{3(s+1)(s-2) - 4(s-2) + (s+1)}{3(s+1)(s-2)} = \frac{3s^2 - 6s + 3}{3(s+1)(s-2)} \\ &= \frac{s^2 - 2s + 1}{(s+1)(s-2)} = \frac{(s-1)^2}{(s+1)(s-2)} \quad \text{if } \; Re\{s\} > 2. \end{aligned}$$



Inverse Laplace Transform by Partial Fraction Expansion

Section 9.3 *in* Oppenheim & Willsky

<u>Example 8:</u>

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{(A+B)s + (2A+B)}{(s+1)(s+2)}$$

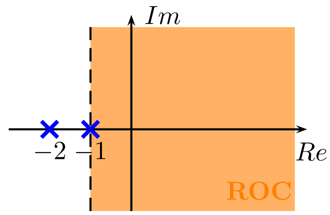
$$\left.\begin{aligned} A + B &= 0 \\ 2A + B &= 1 \end{aligned}\right\} \quad \begin{aligned} A &= 1 \\ B &= -1 \end{aligned}$$

Note that:

$$\frac{1}{s+1} \begin{array}{l} \rightarrow \; e^{-t}u(t), \; Re\{s\} > -1 \\ \searrow \; -e^{-t}u(-t), \; Re\{s\} < -1 \end{array} \qquad \frac{1}{s+2} \begin{array}{l} \rightarrow \; e^{-2t}u(t), \; Re\{s\} > -2 \\ \searrow \; -e^{-2t}u(-t), \; Re\{s\} < -2 \end{array}$$

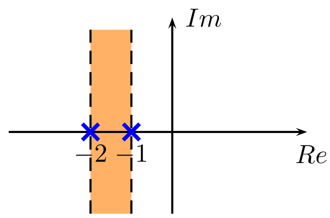
Thus, $x(t)$ can't be determined uniquely unless the ROC is specified.

Possibilities:
1) $x(t) = e^{-t}u(t) + e^{-2t}u(t)$, if $Re\{s\} > -1$.

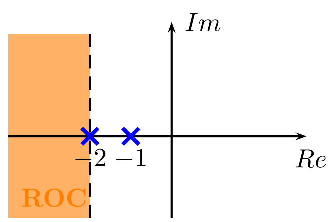


2) $x(t) = e^{-t}u(t) - e^{-2t}u(-t)$, $\underline{ROC = \emptyset}$
since $Re\{s\} > -1$ and $Re\{s\} < -2$ do not intersect.

3) $x(t) = -e^{-t}u(-t) + e^{-2t}u(t)$ if $-2 < Re\{s\} < -1$



4) $x(t) = -e^{-t}u(-t) - e^{-2t}u(-t)$ if $Re\{s\} < -2$



Properties of the Laplace Transform

Section 9.5 *in* Oppenheim & Willsky

Assume that $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with $ROC = R$.

<u>Linearity:</u>

$$ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\longleftrightarrow} aX_1(s) + bX_2(s) \tag{6}$$

ROC contains $R_1 \cap R_2$, but can be larger: *e.g.*, if $x_1(t) = x_2(t)$ and $a = -b$, then $ax_1(t) + bx_2(t) \equiv 0$ and ROC is the entire complex plane.

Time-Shift:

$$\boxed{x(t-t_0) \leftrightarrow e^{-st_0}X(s)} \tag{7}$$

ROC unchanged because:

$$\int_{-\infty}^{\infty} x(\underbrace{t-t_0}_{\triangleq\tau})e^{-st}dt = \int_{-\infty}^{\infty} x(\tau)e^{-s\tau}e^{-st_0}d\tau = \underbrace{e^{-st_0}}_{\substack{\text{this factor}\\\text{doesn't change}\\\text{convergence}}} \underbrace{\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau}_{X(s)}$$

Shifting in the s-Domain:

$$\boxed{e^{s_0t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-s_0) \quad ROC = R + Re\{s_0\}} \tag{8}$$

Compare to: $e^{j\omega_0 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega-\omega_0))$ in Fourier transforms.

Time-Scaling:

$$\boxed{x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right) \quad ROC = a \cdot R} \tag{9}$$

In particular, $x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s)$, with $ROC = -R$.

Example 9:

$$\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega_0^2} \qquad Re\{s\} > 0$$

$$\cos(-\omega_0 t)u(-t) = \cos(\omega_0 t)u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-s}{s^2+\omega_0^2} \qquad Re\{s\} < 0$$

Conjugation:

$$\boxed{x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*) \quad ROC \text{ unchanged}} \tag{10}$$

Similar property in Fourier transforms: $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$

Convolution:

$$\boxed{(x_1 * x_2)(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s) \quad ROC \text{ contains } R_1 \cap R_2} \tag{11}$$

Differentiation in Time Domain:

$$\boxed{\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s) \quad ROC \text{ contains } R \text{ but can be larger}} \tag{12}$$

Example 10: $x(t) = \sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{\omega_0}{s^2+\omega_0^2} \qquad Re\{s\} > 0$

$$\frac{dx(t)}{dt} = \omega_0\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0 s}{s^2+\omega_0^2} \qquad Re\{s\} > 0$$

Differentiation in the s-Domain:

$$\boxed{-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds} \quad ROC \text{ unchanged for exponential signals}} \tag{13}$$

Proof: $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$ then $\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} -tx(t)e^{-st}dt$.

Example 11:

$$\begin{aligned}
e^{-at}u(t) &\overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \\
te^{-at}u(t) &\overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2} \\
t^2e^{-at}u(t) &\overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\left(\frac{1}{(s+a)^2}\right) = \frac{2}{(s+a)^3} \\
&\vdots \\
t^ne^{-at}u(t) &\overset{\mathcal{L}}{\longleftrightarrow} \frac{n!}{(s+a)^{n+1}}
\end{aligned}$$

with $Re\{s\} > -a$ for all cases.

Special case $a = 0$: $u(t) \leftrightarrow \frac{1}{s}$, $tu(t) \leftrightarrow \frac{1}{s^2}$, ..., $t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$

Example 12: Partial fraction expansion for repeated poles

Given $ROC = \{s : \ Re\{s\} > -1\}$, find the inverse Laplace transform for:

$$X(s) = \frac{1}{(s+1)(s+2)^2}.$$

$$\begin{aligned}
X(s) &= \frac{1}{(s+1)(s+2)^2} = \frac{A_1}{s+1} + \frac{A_{21}}{s+2} + \frac{A_{22}}{(s+2)^2} \\
&= \frac{A_1(s+2)^2 + A_{21}(s+1)(s+2) + A_{22}(s+1)}{(s+1)(s+2)^2}
\end{aligned}$$

$$\underbrace{(A_1 + A_{21})}_{=0}s^2 + \underbrace{(4A_1 + 3A_{21} + A_{22})}_{=0}s + \underbrace{(4A_1 + 2A_{21} + A_{22})}_{=1} = 1$$

$$\implies A_1 = 1, \quad A_{21} = A_{22} = -1$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} \leftrightarrow x(t) = (e^{-t} - e^{-2t} - te^{-2t})u(t)$$

Integration in Time:

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s) \quad ROC \text{ contains } R \cap \{s : Re\{s\} > 0\}} \tag{14}$$

Follows from the convolution property: $\int_{-\infty}^{t} x(\tau)d\tau = (x * u)(t)$ where $u(t) \leftrightarrow \frac{1}{s}$ is the unit step.

Initial Value Theorem:
If $x(t) = 0$ for all $t < 0$ and contains no impulses or singularities at $t = 0$, then

$$\boxed{x(0^+) = \lim_{s\to\infty} sX(s)} \tag{15}$$

Example 13: $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$

$$\lim_{s\to\infty} \frac{s}{s+a} = 1 = e^{-at}u(t)|_{t=0^+}$$

Final Value Theorem:
If $x(t) = 0$ for all $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)} \tag{16}$$

Signal	Transform	ROC
$\delta(t)$	1	all s
$u(t)$	$\frac{1}{s}$	$Re\{s\} > 0$
$-u(-t)$	$\frac{1}{s}$	$Re\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$Re\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$Re\{s\} < 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re\{s\} > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$Re\{s\} < -a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$Re\{s\} > -a$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$Re\{s\} < -a$
$\delta(t-T)$	e^{-sT}	all s
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re\{s\} > 0$
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$Re\{s\} > 0$
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re\{s\} > -a$
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re\{s\} > -a$

Table 1: Laplace transforms of several functions.